

# ON SEMI-CONTINUITY PROBLEMS FOR MINIMAL LOG DISCREPANCIES

YUSUKE NAKAMURA

**ABSTRACT.** We show the semi-continuity property of minimal log discrepancies for varieties which have a crepant resolution in the category of Deligne-Mumford stacks. Using this property, we also prove the ideal-adic semi-continuity problem for toric pairs.

## 1. INTRODUCTION

The minimal log discrepancy (mld for short) was introduced by Shokurov, in order to reduce the conjecture of terminations of flips to a local problem about singularities. Recently, this has been a fundamental invariant in the minimal model program. There are two related conjectures about mld's, ACC (ascending chain condition) conjecture and LSC (lower semi-continuity) conjecture. Shokurov showed that these two conjectures imply the conjecture of terminations of flips [18].

In the first half of this paper, we consider LSC conjecture, proposed by Ambro [1].

**Conjecture 1.1** (LSC conjecture). *Let  $(X, \Delta)$  be a log pair. Then, the function*

$$|X| \rightarrow \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \text{mld}_x(X, \Delta)$$

*is lower semi-continuous, where we denote by  $|X|$  the set of all closed points of  $X$ .*

LSC conjecture is known under some conditions. If  $X$  is toric or  $\dim X \leq 3$ , this conjecture is known by work of Ambro [2]. If  $X$  is smooth, this conjecture was proved by Ein, Mustařă, and Yasuda [7]. They used the description of mld's by the language of jet schemes. Ein and Mustařă extended this result to locally complete intersection varieties [6].

The first purpose of this paper is to prove LSC conjecture on varieties with quotient singularities.

Let  $X$  be a  $\mathbb{Q}$ -Gorenstein normal variety, and assume that  $X$  has a crepant resolution in the category of Deligne-Mumford stacks. Here, a proper birational morphism  $f : \mathcal{X} \rightarrow X$  from a smooth Deligne-Mumford stack  $\mathcal{X}$  is *crepant* if  $K_{\mathcal{X}} = f^*K_X$  holds.

**Theorem 1.2.** *Let  $X$  be a  $\mathbb{Q}$ -Gorenstein normal variety. Assume  $X$  has a crepant resolution in the category of Deligne-Mumford stacks. Then, the followings hold.*

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- (i) Let  $\Delta$  be an effective  $\mathbb{R}$ -Cartier divisor on  $X$ , and  $\mathfrak{a}$  an  $\mathbb{R}$ -ideal sheaf on  $X$ . Then the function

$$|X| \rightarrow \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \text{mld}_x(X, \Delta, \mathfrak{a})$$

is lower semi-continuous.

- (ii) Let  $T$  be a variety,  $\Delta$  an effective  $\mathbb{R}$ -Cartier divisor on  $X$ , and  $x$  a closed point of  $X$ . For an  $\mathbb{R}$ -ideal sheaf  $\mathfrak{A}$  on  $X \times T$ , the function

$$|T| \rightarrow \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad p \mapsto \text{mld}_x(X, \Delta, \mathfrak{A}_p)$$

is lower semi-continuous, where  $\mathfrak{A}_p$  is the restriction of  $\mathfrak{A}$  to  $X \times \{p\}$ .

Especially, Conjecture 1.1 holds for the variety  $X$ .

Assume that a finite group  $G$  acts on a smooth variety  $M$ , and that this action is free in codimension 1. Then the quotient variety  $M/G$  and the quotient stack  $[M/G]$  are isomorphic in codimension 1. Hence,  $M/G$  has a crepant resolution  $[M/G] \rightarrow M/G$ . Therefore we get the following corollary.

**Corollary 1.3.** *Let  $M$  be a smooth variety and  $G$  a finite group. Assume  $G$  acts on  $M$  freely in codimension 1 and the quotient variety  $M/G$  is  $\mathbb{Q}$ -Gorenstein. Then, for any  $\mathbb{R}$ -Cartier divisor  $\Delta$  on  $M/G$ , Conjecture 1.1 holds for the pair  $(M/G, \Delta)$ .*

To prove Theorem 1.2, we employ Yasuda's theory of twisted jet stacks [19], [20]. We take a crepant resolution  $f : \mathcal{X} \rightarrow X$ , and describe  $\text{mld}_x$  by the dimensions of twisted jet stacks on  $\mathcal{X}$ .

As a corollary, we have an application to terminations. We can show that there is no infinite sequence of flips if we start from a symplectic variety with only quotient singularities. This is an extension of Matsushita's result [15, Proposition 2.1, Lemma 2.1].

**Corollary 1.4.** *Let  $X$  be a symplectic variety with only terminal and quotient singularities, and  $D_0$  an effective  $\mathbb{R}$ -Cartier divisor on  $X$ . Then, there is no infinite sequence of flops*

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

such that  $f_i$  is a  $D_i$ -flop, where  $D_i$  is the  $\mathbb{R}$ -Cartier divisor defined as  $D_{i+1} := f_i(D_i)$  inductively.

Especially, if the linear system  $|D_0|$  has no fixed divisor, the  $D$ -MMP terminates.

In the latter half of this paper, we consider the following related conjecture proposed by Mustařă.

**Conjecture 1.5** (Mustařă). *Let  $(X, \Delta)$  be a log pair,  $Z$  a closed subset of  $X$ , and  $I_Z$  its ideal sheaf. Let  $\mathfrak{a} = \prod_{i=1}^s \mathfrak{a}_i^{r_i}$  be an  $\mathbb{R}$ -ideal sheaf with ideal sheaves  $\mathfrak{a}_1, \dots, \mathfrak{a}_s$  on  $X$  and positive real numbers  $r_1, \dots, r_s$ .*

*Then there exists a positive integer  $l$  such that the following holds: if ideal sheaves  $\mathfrak{b}_1, \dots, \mathfrak{b}_s$  satisfy  $\mathfrak{a}_i + I_Z^l = \mathfrak{b}_i + I_Z^l$ , then*

$$\text{mld}_Z(X, \Delta, \mathfrak{a}) = \text{mld}_Z(X, \Delta, \mathfrak{b})$$

*holds, where we put  $\mathfrak{b} := \prod_{i=1}^s \mathfrak{b}_i^{r_i}$ .*

This conjecture is related to Shokurov's ACC conjecture on mld's. In fact, by the method of generic limits, ACC conjecture for locally complete intersection varieties follows from Mustařă's conjecture [9]. Generic limits were introduced by Kollár [13], and using this method, de Fernex, Ein, and Mustařă proved Shokurov's ACC conjecture for log canonical thresholds [3]. The generic limit is one of constructions of a limit of a sequence of ideals. This idea of considering the limit of ideals originates in de Fernex and Mustařă [4], and they constructed a limit using non-standard analysis.

Mustařă's conjecture is known under some conditions. If  $\text{mld}_Z(X, \Delta, \mathfrak{a}) = 0$  holds, the conjecture is known by work of de Fernex, Ein, and Mustařă [3]. If the triple  $(X, \Delta, \mathfrak{a})$  is Kawamata log terminal around  $Z$ , then the conjecture is known by work of Kawakita [9]. The conjecture in dimension 2 was proved by Kawakita [11].

The second purpose of this paper is to prove Mustařă's conjecture on varieties with a  $\mathbb{C}^*$ -action. Let  $X = \text{Spec } A$  be an affine variety with a  $\mathbb{C}^*$ -action, and  $A = \bigoplus_{m \in \mathbb{Z}} A^{(m)}$  the induced graded ring structure. Then, the action of  $\mathbb{C}^*$  on  $X = \text{Spec } A$  is said to be *of ray type* if  $A^{(m)} = 0$  holds for all  $m < 0$  or  $A^{(m)} = 0$  holds for all  $m > 0$ . In the following proposition, we assume that the action is of ray type.

**Proposition 1.6.** *Let  $X$  be a  $\mathbb{Q}$ -Gorenstein normal affine variety,  $\Delta$  an effective  $\mathbb{R}$ -Cartier divisor,  $\mathfrak{a} = \prod_{i=1}^s \mathfrak{a}_i^{r_i}$  be an  $\mathbb{R}$ -ideal sheaf on  $X$ , and  $Z$  a closed subset of  $X$ . Suppose that  $\mathbb{C}^*$  acts on  $X$  and assume the following conditions:*

- *The variety  $X$  has a  $\mathbb{C}^*$ -equivariant crepant resolution in the category of Deligne-Mumford stacks.*
- *The  $\mathbb{C}^*$ -action on  $X$  is of ray type.*
- *The all components of  $\Delta$  and  $\mathfrak{a}_i$  are  $\mathbb{C}^*$ -invariant.*
- *$Z$  is the set of all  $\mathbb{C}^*$ -fixed points in  $X$ .*

*Then, Conjecture 1.5 holds for  $(X, \Delta, \mathfrak{a})$  and  $Z$ .*

*Remark 1.7.* In the above proposition, we assume that the ideal  $\mathfrak{a}_i$  is  $\mathbb{C}^*$ -invariant, but the ideal  $\mathfrak{b}_i$  is not necessarily  $\mathbb{C}^*$ -invariant.

As an application, we can prove Mustařă's conjecture for toric varieties.

**Theorem 1.8.** *Let  $X$  be a normal toric variety,  $(X, \Delta)$  a log pair, and  $Z$  a closed subset of  $X$ . Assume that  $\Delta$  and  $Z$  are torus invariant. Then, for an  $\mathbb{R}$ -ideal sheaf  $\mathfrak{a} = \prod_{i=1}^s \mathfrak{a}_i^{r_i}$  with torus invariant ideal sheaves  $\mathfrak{a}_i$ , Conjecture 1.5 holds for  $(X, \Delta, \mathfrak{a})$  and  $Z$ .*

*Remark 1.9.* As Remark 1.7, in the above theorem, we assume that the ideal  $\mathfrak{a}_i$  is torus invariant, but the ideal  $\mathfrak{b}_i$  in Conjecture 1.5 is not necessarily torus invariant. Therefore the corollary cannot follow directly from a combinatorial description of mld's for toric triples.

In Proposition 1.6, the inequality

$$\text{mld}_Z(X, \Delta, \mathfrak{a}) \leq \text{mld}_Z(X, \Delta, \mathfrak{b})$$

is essential (see Remark 4.2). To prove this inequality, we consider a degeneration of the ideal  $\mathfrak{b}$ . We explain the idea of the proof below. For the sake of

simplicity, we assume  $s = 1$  and denote  $\mathfrak{a}' := \mathfrak{a}_1$ ,  $\mathfrak{b}' := \mathfrak{b}_1$ , and  $r := r_1$ . When  $l$  is sufficiently large, we can degenerate  $\mathfrak{b}'$  to some ideal  $\mathfrak{b}'_0$  which contains  $\mathfrak{a}'$ . Hence we get  $\text{mld}_Z(X, \Delta, (\mathfrak{a}')^r) \leq \text{mld}_Z(X, \Delta, (\mathfrak{b}'_0)^r)$ , and the proposition reduces to the inequality  $\text{mld}_Z(X, \Delta, (\mathfrak{b}'_0)^r) \leq \text{mld}_Z(X, \Delta, (\mathfrak{b}')^r)$ . This inequality follows from the semi-continuity property like Theorem 1.2 (ii). However, the above inequality does not directly follow from Theorem 1.2 (ii). It is because  $Z$  has possibly positive dimension. Hence, we need to look the construction of the above degeneration.

**Notation and convention.** Throughout this paper, we work over the field of complex numbers  $\mathbb{C}$ .

- We denote by  $\mathbb{N}$  the set of all non-negative integers.
- For a Deligne-Mumford stack  $\mathcal{X}$ , we denote by  $|\mathcal{X}|$  the set of all  $\mathbb{C}$ -valued points.
- For a morphism  $f : \mathcal{X} \rightarrow T$  from a Deligne-Mumford stack to a variety  $T$ , and a closed point  $p \in |T|$ , we denote by  $\mathcal{X}_p$  the fiber of  $f$  over  $p$ . For an ideal sheaf  $\mathfrak{a}$  on  $\mathcal{X}$ , we denote by  $\mathfrak{a}_p$  the restriction to the fiber  $\mathcal{X}_p$ . In addition, for a morphism  $g : \mathcal{X} \rightarrow \mathcal{Y}$  between Deligne-Mumford stacks over  $T$ , we denote by  $g_p : \mathcal{X}_p \rightarrow \mathcal{Y}_p$  the induced morphism between the fibers.

## 2. PRELIMINARIES

**2.1. Minimal log discrepancies.** We recall the notations in the theory of singularities in the minimal model program.

A *log pair*  $(X, \Delta)$  is a normal variety  $X$  and an effective  $\mathbb{R}$ -divisor  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. An  *$\mathbb{R}$ -ideal sheaf* of  $X$  is a formal product  $\mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_s^{r_s}$ , where  $\mathfrak{a}_1, \dots, \mathfrak{a}_s$  are ideal sheaves on  $X$  and  $r_1, \dots, r_s$  are positive real numbers. For a log pair  $(X, \Delta)$  and an  $\mathbb{R}$ -ideal sheaf  $\mathfrak{a}$ , we call  $(X, \Delta, \mathfrak{a})$  a *log triple*. When  $\Delta = 0$ , we sometimes drop  $\Delta$  and write  $(X, \mathfrak{a})$ . If  $Y_i$  is the closed subscheme of  $X$  corresponding to  $\mathfrak{a}_i$ , we sometimes identify the triple  $(X, \Delta, \sum_{i=1}^s r_i Y_i)$  with the triple  $(X, \Delta, \prod_{i=1}^s \mathfrak{a}_i^{r_i})$ .

An *extraction* is a proper birational morphism of normal varieties. For an extraction  $f : X' \rightarrow X$  and a prime divisor  $E$  on  $X'$ , the *log discrepancy* of  $(X, \Delta, \mathfrak{a})$  at  $E$  is defined as

$$a_E(X, \Delta, \mathfrak{a}) := 1 + \text{ord}_E(K_{X'} - f^*(K_X + \Delta)) - \text{ord}_E \mathfrak{a},$$

where  $\text{ord}_E \mathfrak{a} := \sum_{i=1}^s r_i \text{ord}_E \mathfrak{a}_i$ . The image  $f(E)$  is called the *center of  $E$  on  $X$* , and we denote it by  $c_X(E)$ . For a closed subset  $Z$  of  $X$ , the *minimal log discrepancy* (mld for short) over  $Z$  is defined as

$$\text{mld}_Z(X, \Delta, \mathfrak{a}) := \inf_{c_X(E) \subset Z} a_E(X, \Delta, \mathfrak{a}),$$

where the infimum is taken after all prime divisors on extractions of  $X$  with the center  $c_X(E) \subset Z$ .

*Remark 2.1.* It is known that  $\text{mld}_Z(X, \Delta, \mathfrak{a})$  is in  $\mathbb{R}_{\geq 0} \cup \{-\infty\}$  and that if  $\text{mld}_Z(X, \Delta, \mathfrak{a}) \geq 0$ , then the infimum on right hand side in the definition is actually the minimum.

*Remark 2.2.* Let  $(X, \Delta, \mathfrak{a})$  and  $Z$  be as above. Mld's have the following properties.

(i) If  $Z_1, \dots, Z_c$  are the irreducible components of  $Z$ , then

$$\mathrm{mld}_Z(X, \Delta, \mathbf{a}) = \min_{1 \leq i \leq c} \mathrm{mld}_{Z_i}(X, \Delta, \mathbf{a}).$$

(ii) If  $U_1, \dots, U_c$  are open subsets of  $X$  such that  $Z \subset \bigcup_{j=1}^c U_j$ , then

$$\mathrm{mld}_Z(X, \Delta, \mathbf{a}) = \min_{1 \leq j \leq c} \mathrm{mld}_{Z \cap U_j}(U_j, \Delta|_{U_j}, \mathbf{a}|_{U_j}),$$

$$\text{where } \mathbf{a}|_{U_j} := \prod_{i=1}^s (\mathbf{a}_i \mathcal{O}_{U_j})^{r_i}.$$

These properties easily follow from the definition.

**2.2. Notation and remarks on Deligne-Mumford stacks.** In this section, we review some properties of Deligne-Mumford stacks (DM stacks for short). In this paper, we are mainly interested in separated DM stacks of finite type over complex number  $\mathbb{C}$ .

Let  $\mathcal{X}$  be a DM stack of finite type over  $\mathbb{C}$ . We can consider the set of *points*  $|\mathcal{X}|$  of  $\mathcal{X}$  over  $\mathbb{C}$ , and it admits a Zariski topology [14]. Keel and Mori [12] proved that the coarse moduli space  $X$  exists for  $\mathcal{X}$ . Then, the induced map  $|\mathcal{X}| \rightarrow |X|$  is a homeomorphism.

It is known that a DM stack is étale locally isomorphic to a quotient stack (see for instance [12]). Let  $X$  be the coarse moduli space of  $\mathcal{X}$ . Then, there exists an étale covering  $\{X_i \rightarrow X\}_i$  such that for every  $i$ , the étale pullback  $\mathcal{X} \times_X X_i$  is isomorphic to a quotient stack  $[M_i/G_i]$ , where  $M_i$  is a variety over  $\mathbb{C}$  and  $G_i$  is a finite group.

We prove two lemmas about DM stacks. First one is about families of DM stacks.

**Lemma 2.3.** *Let  $M$  be a variety and  $G$  a finite group. Suppose  $G$  acts on  $M$ . Let  $T$  be a variety and  $f : [M/G] \rightarrow T$  be a morphism of stacks. Then, for a closed point  $p \in |T|$ , the fiber  $[M/G]_p$  is isomorphic to the quotient stack  $[M_p/G]$ .*

*Proof.* First, we remark that  $M_p$  is  $G$ -invariant because the morphism  $M \rightarrow T$  factors through the quotient stack  $[M/G]$ .

Let  $n$  be the order of the group  $G$ . Then, both of the quotient maps  $M \rightarrow [M/G]$  and  $M_p \rightarrow [M_p/G]$  are surjective finite étale morphisms of degree  $n$ . It follows that both of the morphisms  $M_p \rightarrow [M/G]_p$  and  $M_p \rightarrow [M_p/G]$  are surjective finite étale morphisms of degree  $n$ . Since  $M_p \rightarrow [M/G]_p$  factors through  $M_p \rightarrow [M_p/G]$ , we can conclude that  $[M/G]_p$  is isomorphic to  $[M_p/G]$ .  $\square$

Second lemma is about semi-continuity of the dimensions of fibers on a family of DM stacks.

**Lemma 2.4.** *Let  $\mathcal{X}$  be a DM stack of finite type over  $\mathbb{C}$ , and  $f : \mathcal{X} \rightarrow T$  a morphism from  $\mathcal{X}$  to a variety  $T$ . Then, for every  $n \in \mathbb{N}$ , the set*

$$|\mathcal{X}|_{\geq n} := \{x \in |\mathcal{X}| \mid \dim_x f^{-1}(f(x)) \geq n\}$$

*is closed in  $|\mathcal{X}|$ . In the above equation, we denote by  $\dim_x$  the dimension around  $x$ .*

*Proof.* If  $\mathcal{X}$  is a variety, then this lemma is well-known (see for instance [8]).

Let  $X$  be the coarse moduli space of  $\mathcal{X}$ . By the universality of the coarse moduli space,  $f : \mathcal{X} \rightarrow T$  factors through  $\mathcal{X} \rightarrow X$ . Since the induced

map  $|\mathcal{X}| \rightarrow |X|$  is a homeomorphism, the assertion follows from the case of varieties.  $\square$

**2.3. Yasuda's twisted jet stacks.** Twisted jet stacks were introduced by Yasuda [19], [20]. First, we recall the definition.

Let  $\mathcal{X}$  be a DM stack over  $\mathbb{C}$ . For an affine scheme  $S = \operatorname{Spec} R$  over  $\mathbb{C}$  and a non-negative integer  $n$ , we denote

$$D_{n,S} := \operatorname{Spec} R[t]/(t^{n+1}).$$

For a positive integer  $l$ , we denote by  $\mu_l$  the cyclic group of  $l$ -th roots of the unity. We consider the natural group action  $\mu_l$  on  $D_{nl,S}$  induced by the following ring homomorphism:

$$R[t]/(t^{nl+1}) \rightarrow R[t]/(t^{nl+1}) \otimes \mathbb{C}[x]/(x^l - 1); \quad t \mapsto t \otimes x.$$

Then, we denote

$$\mathcal{D}_{n,S}^l := [D_{nl,S}/\mu_l].$$

A *twisted  $n$ -jet of order  $l$*  of  $\mathcal{X}$  over  $S$  is a representable morphism  $\mathcal{D}_{n,S}^l \rightarrow \mathcal{X}$ . Yasuda defined the *stack of twisted  $n$ -jets of order  $l$* , and we denote it by  $\mathcal{J}_n^l \mathcal{X}$ . An object of  $\mathcal{J}_n^l \mathcal{X}$  over  $S$  is a twisted  $n$ -jet of order  $l$ . For a morphism  $f : S \rightarrow T$  over  $\mathbb{C}$ , we have an induced morphism  $f' : \mathcal{D}_{n,S}^l \rightarrow \mathcal{D}_{n,T}^l$ . A morphism in  $\mathcal{J}_n^l \mathcal{X}$  from  $\gamma : \mathcal{D}_{n,S}^l \rightarrow \mathcal{X}$  to  $\gamma' : \mathcal{D}_{n,T}^l \rightarrow \mathcal{X}$  is a 2-morphism from  $\gamma$  to  $\gamma' \circ f'$ . The stack  $\mathcal{J}_n \mathcal{X}$  of *twisted  $n$ -jets* is the disjoint union of the stacks  $\bigsqcup_{l \geq 1} \mathcal{J}_n^l \mathcal{X}$ . Both  $\mathcal{J}_n^l \mathcal{X}$  and  $\mathcal{J}_n \mathcal{X}$  are actually DM stacks [20, Theorem 18]. The stack  $\mathcal{J}_0 \mathcal{X}$  can be identified with the inertia stack and the projection

$$\varphi_{0,b}^{\mathcal{X}} : \mathcal{J}_0 \mathcal{X} \rightarrow \mathcal{X}$$

is a finite morphism. When  $X$  is a scheme,  $\mathcal{J}_n^l X = \emptyset$  for  $l \geq 2$ , and  $\mathcal{J}_n X$  can be identified with the usual  $n$ -th jet scheme  $J_n X$ .

For  $0 \leq n_1 \leq n_2$ , we have the *truncation morphism*  $\varphi_{n_2,n_1}^{\mathcal{X}} : \mathcal{J}_{n_2} \mathcal{X} \rightarrow \mathcal{J}_{n_1} \mathcal{X}$ . This corresponds to the surjective ring homomorphism  $R[t]/(t^{n_1 l+1}) \rightarrow R[t]/(t^{n_2 l+1})$ . Since  $\varphi_{n_2,n_1}^{\mathcal{X}}$  is an affine morphism, we have a projective limit and projections

$$\mathcal{J}_{\infty} \mathcal{X} := \varprojlim \mathcal{J}_n \mathcal{X}, \quad \varphi_{\infty,n}^{\mathcal{X}} : \mathcal{J}_{\infty} \mathcal{X} \rightarrow \mathcal{J}_n \mathcal{X}.$$

In the theory of jet schemes, the  $\mathbb{C}^*$ -action and the relative jet schemes can be defined [16, Section 2]. In the following part of this section, we generalize these concepts to twisted jet stacks.

We already defined the truncation morphism  $\varphi_{n,0}^{\mathcal{X}} : \mathcal{J}_n \mathcal{X} \rightarrow \mathcal{J}_0 \mathcal{X}$ . On the other hand, we also have the *zero section*  $\sigma_n^{\mathcal{X}} : \mathcal{J}_0 \mathcal{X} \rightarrow \mathcal{J}_n \mathcal{X}$ . This is defined by the composition with the stack morphism  $\mathcal{D}_{n,S}^l \rightarrow \mathcal{D}_{0,S}^l$  induced by the ring inclusion  $R \hookrightarrow R[t]/(t^{nl+1})$ . We will write simply  $\varphi_{m,n}$  (resp.  $\sigma_n$ ) instead of  $\varphi_{m,n}^{\mathcal{X}}$  (resp.  $\sigma_n^{\mathcal{X}}$ ) when no confusion can arise.

The twisted jet stack  $\mathcal{J}_n \mathcal{X}$  admits the  $\mathbb{C}^*$ -action which is induced by the following  $\mathbb{C}^*$ -action on  $R[t]/(t^{nl+1})$ :

$$R[t]/(t^{nl+1}) \rightarrow R[t]/(t^{nl+1}) \otimes \mathbb{C}[s, s^{-1}]; \quad t \mapsto t \otimes s.$$

The following lemma is used in the proof of Theorem 1.2.

**Lemma 2.5.** *Let  $W$  be a  $\mathbb{C}^*$ -invariant closed subset of  $|\mathcal{J}_n\mathcal{X}|$ . Then  $\varphi_{n,0}(W)$  is a closed subset of  $|\mathcal{J}_0\mathcal{X}|$ .*

*Proof.* The  $\mathbb{C}^*$ -action on  $\mathcal{J}_n\mathcal{X}$  is induced by the action  $\mathbb{C}^* \times \mathcal{D}_{n,S}^l \rightarrow \mathcal{D}_{n,S}^l$ . By the definition above, the morphism  $\mathbb{C}^* \times \mathcal{D}_{n,S}^l \rightarrow \mathcal{D}_{n,S}^l$  is uniquely extended to the morphism  $\mathbb{C} \times \mathcal{D}_{n,S}^l \rightarrow \mathcal{D}_{n,S}^l$ . Therefore, the  $\mathbb{C}^*$ -action  $\mathbb{C}^* \times \mathcal{J}_n\mathcal{X} \rightarrow \mathcal{J}_n\mathcal{X}$  is extended to the morphism

$$\psi : \mathbb{C} \times \mathcal{J}_n\mathcal{X} \rightarrow \mathcal{J}_n\mathcal{X}.$$

By definition, for any  $\alpha \in |\mathcal{J}_n\mathcal{X}|$ , we have  $\psi(0, \alpha) = \sigma_n(\varphi_{n,0}(\alpha))$ .

It is sufficient to show the equality  $\varphi_{n,0}(W) = \sigma_n^{-1}(W)$  because  $\sigma_n^{-1}(W)$  is closed. Fix an element  $\alpha \in W$ . Since  $W$  is  $\mathbb{C}^*$ -invariant,  $\psi(\gamma, \alpha) \in W$  for any  $\gamma \in \mathbb{C}^*$ . Because  $W$  is closed,  $\psi(0, \alpha) \in W$  holds. Since  $\psi(0, \alpha) = \sigma_n(\varphi_{n,0}(\alpha))$ , we have the equality  $\varphi_{n,0}(W) = \sigma_n^{-1}(W)$ .  $\square$

Next, we construct the *relative twisted jet stacks*.

**Lemma 2.6.** *Let  $f : \mathcal{Y} \rightarrow T$  be a morphism from a DM stack  $\mathcal{Y}$  to a variety  $T$ . Then there exist a DM stack  $\mathcal{J}_n(\mathcal{Y}/T)$  and a morphism  $g : \mathcal{J}_n(\mathcal{Y}/T) \rightarrow T$  such that for any point  $p \in |T|$ ,*

$$\mathcal{J}_n(\mathcal{Y}/T)_p \cong \mathcal{J}_n(\mathcal{Y}_p)$$

*holds, where  $\mathcal{J}_n(\mathcal{Y}/T)_p$  is the fiber of  $g$  over  $p$ , and  $\mathcal{Y}_p$  is the fiber of  $f$  over  $p$ .*

For the proof of Lemma 2.6, we use an étale local description of twisted jet stacks [20, Proposition 16].

Let  $[M/G]$  be a quotient stack with scheme  $M$  and finite group  $G$ . Fix a positive integer  $l$  and an embedding  $a : \mu_l \hookrightarrow G$ . We denote by  $J_n M$  the  $n$ -th jet scheme of  $M$ , then  $\mu_l$  acts on  $J_n M$  in two ways. First, the action  $\mu_l \curvearrowright D_{n,S}$  induces an action  $\mu_l \curvearrowright J_n M$ . On the other hand, we have the action  $G \curvearrowright J_n M$ , and this action induces an action  $\mu_l \curvearrowright J_n M$  by the embedding  $a : \mu_l \hookrightarrow G$ . We define  $J_n^{(a)} M$  to be the closed subscheme of  $J_n M$  where the above two actions are identical. Then we have a concrete description of twisted jet stacks:

$$\mathcal{J}_n^l \mathcal{X} \cong \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_n^{(a)} M / C_a],$$

where  $\text{Conj}(\mu_l, G)$  is the conjugacy classes of embeddings  $\mu_l \hookrightarrow G$ , and  $C_a$  is the centralizer of  $a$ .

*Proof of Lemma 2.6.* When  $\mathcal{Y} = Y$  is a scheme, the *relative jet scheme*  $J_n(Y/T)$  exists and

$$J_n(Y/T)_p \cong J_n(Y_p)$$

holds [16, Section 2]. Besides,  $J_n(Y/T)$  can be characterized by the following representability:

$$\text{Hom}_{\text{Sch}/T}(Z, J_n(Y/T)) \cong \text{Hom}_{\text{Sch}/T}(Z \times_{\mathbb{C}} \text{Spec}(\mathbb{C}[t]/(t^{n+1})), Y)$$

for every scheme  $Z$  over  $T$ .

Since the problem is étale local on  $T$ , we may assume that  $\mathcal{Y}$  is a quotient stack  $[M/G]$ . Fix a positive integer  $l$  and an embedding  $a : \mu_l \hookrightarrow$

$G$ . Then, two actions  $\mu_l \curvearrowright J_n(M/T)$  are induced by the actions  $\mu_l \curvearrowright \text{Spec}(\mathbb{C}[t]/(t^{m+1}))$  and  $\mu_l \curvearrowright M$ . These actions are compatible with above two  $\mu_l$ -actions on  $J_n(M_p)$  when we restrict them to the fibers  $J_n(M/T)_p$ . Set  $J_n^{(a)}(M/T)$  to be the closed subscheme of  $J_n(M/T)$  where the above two actions are identical.

We put

$$\mathcal{J}_n^l([M/G]/T) := \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_{nl}^{(a)}(M/T)/C_a].$$

Restricting this to the fiber over  $p \in |T|$ , we have

$$\mathcal{J}_n^l([M/G]/T)_p \cong \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_{nl}^{(a)}(M_p)/C_a] \cong \mathcal{J}_n^l([M/G]_p)$$

by Lemma 2.3. Hence, if we define

$$\mathcal{J}_n([M/G]/T) := \bigsqcup_{l \geq 1} \mathcal{J}_n^l([M/G]/T),$$

$\mathcal{J}_n([M/G]/T)$  satisfies the property in the statement.  $\square$

For the relative twisted jet stacks, we can also define the truncation morphism  $\varphi_{n_2, n_1}^{\mathcal{Y}/T} : \mathcal{J}_{n_2}(\mathcal{Y}/T) \rightarrow \mathcal{J}_{n_1}(\mathcal{Y}/T)$ , the zero section  $\sigma_n^{(\mathcal{Y}/T)} : \mathcal{J}_0(\mathcal{Y}/T) \rightarrow \mathcal{J}_n(\mathcal{Y}/T)$ , and  $\mathbb{C}^*$ -action on  $\mathcal{J}_n(\mathcal{Y}/T)$ . If we restrict them to the fiber over a closed point  $p \in T$ , these definitions are compatible with the definitions in the absolute case. We also have the following lemma.

**Lemma 2.7.** *Let  $W$  be a  $\mathbb{C}^*$ -invariant closed subset of  $|\mathcal{J}_n(\mathcal{Y}/T)|$ . Then  $\varphi_{n,0}^{\mathcal{Y}/T}(W)$  is a closed subset of  $|\mathcal{J}_0(\mathcal{Y}/T)|$ .*

*Proof.* We have the equality  $\varphi_{n,0}^{\mathcal{Y}/T}(W) = (\sigma_n^{\mathcal{Y}/T})^{-1}(W)$  by the proof of Lemma 2.5. The right hand side is clearly closed in  $|\mathcal{J}_0(\mathcal{Y}/T)|$ .  $\square$

**2.4. Motivic integration.** First, we define the space in which motivic integration takes value. We introduce the notion of the *Grothendieck semiring*, following Yasuda [20, Section 3].

In this section, we fix a positive integer  $r$ . A *convergent stack* is the pair  $(\mathcal{X}, \alpha)$  of a DM stack  $\mathcal{X}$  of finite type and a function

$$\alpha : \{\text{connected component of } \mathcal{X}\} \rightarrow \frac{1}{r}\mathbb{Z}$$

satisfying the following two conditions:

- $\mathcal{X}$  has at most countably many connected components,
- for every integer  $n$ , there are at most finitely many connected components  $\mathcal{V}$  such that  $\dim \mathcal{V} + \alpha(\mathcal{V}) > n$ .

A DM stack  $\mathcal{X}$  of finite type is identified with the convergent stack  $(\mathcal{X}, 0)$ . For two convergent stacks  $(\mathcal{X}, \alpha)$  and  $(\mathcal{Y}, \beta)$ , a *morphism*  $f : (\mathcal{X}, \alpha) \rightarrow (\mathcal{Y}, \beta)$  of convergent stacks is a morphism  $g : \mathcal{X} \rightarrow \mathcal{Y}$  of stacks satisfying  $\beta = \alpha \circ f$ . We denote by  $(\mathfrak{X}^{1/r})'$  the set of the isomorphism classes of convergent stacks.  $(\mathfrak{X}^{1/r})'$  admits a semiring structure by the disjoint union  $\sqcup$  and the product  $\times$ . For  $(\mathcal{X}, \alpha) \in (\mathfrak{X}^{1/r})'$ , we can define the *dimension*  $\dim(\mathcal{X}, \alpha)$  by

$$\max\{\dim \mathcal{V} + \alpha(\mathcal{V}) \mid \mathcal{V} \text{ is a connected component of } \mathcal{X}\}.$$



For empty set  $\emptyset$ , we define  $\dim \emptyset := -\infty$ .

For each  $n \in \mathbb{Z}$ , we can define a relation  $\sim_n$  on  $(\mathfrak{R}^{1/r})'$  to be the strongest equivalence relation satisfying the following three relations:

- If  $(\mathcal{X}, \alpha)$  and  $(\mathcal{Y}, \beta)$  are convergent stacks with  $\dim(\mathcal{Y}, \beta) < n$ , then  $(\mathcal{X}, \alpha) \sim_n (\mathcal{X}, \alpha) \sqcup (\mathcal{Y}, \beta)$ .
- If  $(\mathcal{X}, \alpha)$  is a convergent stack and  $\mathcal{Y}$  is a closed substack of  $\mathcal{X}$ , then  $(\mathcal{X}, \alpha) \sim_n (\mathcal{Y}, \alpha|_{\mathcal{Y}}) \sqcup (\mathcal{X} \setminus \mathcal{Y}, \alpha)$ .
- Let  $(\mathcal{X}, \alpha)$  and  $(\mathcal{Y}, \beta)$  be convergent stacks. Assume there exists a representable morphism of stacks  $f : \mathcal{X} \rightarrow \mathcal{Y}$  such that  $f^{-1}(x) \cong \mathbb{A}^{\beta(x) - \alpha(f^{-1}(x))}$  holds for any point  $x \in |\mathcal{Y}|$ . Then  $(\mathcal{X}, \alpha) \sim_n (\mathcal{Y}, \beta)$ .

Then we can define new relation  $\sim$  on  $(\mathfrak{R}^{1/r})'$  as follows: For any  $a, b \in (\mathfrak{R}^{1/r})'$ ,  $a \sim b$  if and only if  $a \sim_n b$  for any integer  $n$ . For a convergent stack  $(\mathcal{X}, \alpha) \in (\mathfrak{R}^{1/r})'$ , we denote by  $\{(\mathcal{X}, \alpha)\}$  the equivalence classes of  $\mathcal{X}$ . Besides, we denote by  $\mathfrak{R}^{1/r}$  the set of all the equivalence classes. By definition, the map

$$\dim : \mathfrak{R}^{1/r} \rightarrow \frac{1}{r}\mathbb{Z} \cup \{-\infty\}, \quad \{(\mathcal{X}, \alpha)\} \mapsto \dim(\mathcal{X}, \alpha)$$

is well defined, and a semiring structure on  $\mathfrak{R}^{1/r}$  is induced by the semiring structure on  $(\mathfrak{R}^{1/r})'$ .

Next, we introduce the motivic measure on a smooth DM stack [20, Section 2].

Let  $\mathcal{X}$  be a smooth DM stack of finite type and pure dimension  $d$ . For a non-negative integer  $n$ , a subset  $A \subset |\mathcal{J}_\infty \mathcal{X}|$  is called an  $n$ -cylinder if  $A = \varphi_{\infty, n}^{-1} \varphi_{\infty, n}(A)$  and  $\varphi_{\infty, n}(A)$  is a constructible subset of  $|\mathcal{J}_n \mathcal{X}|$ . For an  $n$ -cylinder  $A \subset |\mathcal{J}_\infty \mathcal{X}|$  we define

$$\mu_{\mathcal{X}}(A) := \{\varphi_{\infty, n}(A)\} \mathbb{L}^{-nd} \in \mathfrak{R}^{1/r},$$

where we denote  $\{\mathbb{A}^1\}$  by  $\mathbb{L}$ . This definition is independent of  $n$  [20, Lemma 43]. A subset  $A \subset |\mathcal{J}_\infty \mathcal{X}|$  is called *negligible* if there exist cylinders  $C_i$  for  $i \geq 1$  such that  $A = \bigcap_{i \geq 1} C_i$  and  $\lim_{i \rightarrow \infty} \text{codim } C_i = \infty$  hold, where we denote  $\text{codim } C := \text{codim}_{|\mathcal{J}_n \mathcal{X}|} \varphi_{\infty, n}(C)$  for an  $n$ -cylinder  $C$ .

Let  $A \subset |\mathcal{J}_\infty \mathcal{X}|$  be a subset. A function  $F : A \rightarrow \mathfrak{R}^{1/r}$  is called *measurable* if there exist a negligible subset  $B$  and countably many cylinders  $A_i$  such that  $A = B \sqcup \bigsqcup_{i \geq 1} A_i$  and  $F$  is constant on  $A_i$ . For such  $F$ , we define the *motivic integration* as follows:

$$\int_A F d\mu_{\mathcal{X}} := \sum_{i \geq 1} F(A_i) \cdot \mu_{\mathcal{X}}(A_i) \in \mathfrak{R}^{1/r} \cup \{\infty\},$$

where it takes value  $\infty$  if there exist infinitely many  $i$  with  $\dim F(A_i) \cdot \mu_{\mathcal{X}}(A_i) > m$  for some integer  $m$ .

Next, we define the motivic measure on singular varieties [5], [20].

Let  $X$  be a variety of dimension  $d$ . For a non-negative integer  $n$ , a subset  $A \subset |J_\infty X|$  is called *stable* at level  $n$  if

- $A = \varphi_{\infty, n}^{-1} \varphi_{\infty, n}(A)$ ,
- $\varphi_{\infty, n}(A)$  is a constructible subset of  $|J_n X|$ , and
- for any  $m \geq n$  the truncation morphism  $\varphi_{m, n} : \varphi_{\infty, m}(A) \rightarrow \varphi_{\infty, n}(A)$  is a piecewise trivial fibration with fibers  $\mathbb{A}^{(m-n)d}$ .

For such a subset  $A \subset |J_\infty X|$ , we define

$$\mu_X(A) := \{\varphi_{\infty,n}(A)\} \mathbb{L}^{-nd} \in \mathfrak{R}^{\frac{1}{r}}.$$

A subset  $A \subset |J_\infty X|$  is called *negligible* if there exist constructible subsets  $C_i \subset \varphi_{\infty,i}(J_\infty X)$  for  $i \geq 0$  such that  $A = \bigcap_{i \geq 0} C_i$  and  $\lim_{i \rightarrow \infty} \dim C_i - di = -\infty$  hold. As in the case of smooth DM stacks, we can define the motivic integration on singular varieties, replacing  $n$ -cylinders by stable subsets at level  $n$  in the above definition.

**2.5. Transformation rule.** Yasuda proved the transformation rule for a proper birational morphism from a smooth DM stack to a variety.

First, we define the *shift function* [20].

Let  $\mathcal{X}$  be a smooth DM stack of dimension  $d$ ,  $x \in |\mathcal{X}|$  a  $\mathbb{C}$ -valued point, and  $a : \mu_l \hookrightarrow \text{Aut}(x)$  an embedding. Then the cyclic group  $\mu_l$  acts on the tangent space  $T_x \mathcal{X}$  by the embedding  $a$ , and induces a decomposition  $T_x \mathcal{X} = \bigoplus_{i=1}^l T_{x,i}$ , where  $T_{x,i}$  is the eigenspace on which  $\xi_l := e^{2\pi\sqrt{-1}/l} \in \mu_l$  acts by the multiplication of  $\xi_l^i$ . Then we define

$$\text{sht}(a) := d - \frac{1}{l} \sum_{i=1}^l i \cdot \dim T_{x,i} \in \mathbb{Q}.$$

Since the space  $|\mathcal{J}_0 \mathcal{X}|$  is set theoretically equal to  $\bigsqcup_{l \geq 1} \bigsqcup_{x \in |\mathcal{X}|} \text{Conj}(\mu_l, \text{Aut}(x))$ , we can write a point of  $|\mathcal{J}_0 \mathcal{X}|$  by a pair  $(x, a)$  with  $x \in |\mathcal{X}|$  and  $a \in \text{Conj}(\mu_l, \text{Aut}(x))$ . Then we can define shift function on  $|\mathcal{J}_0 \mathcal{X}|$  by followings

$$\text{sht} : |\mathcal{J}_0 \mathcal{X}| \rightarrow \mathbb{Q}; \quad (x, a) \mapsto \text{sht}(a).$$

This function is constant on each connected component  $\mathcal{V}$  of  $|\mathcal{J}_0 \mathcal{X}|$ . We define shift function  $\mathfrak{s}_{\mathcal{X}}$  on  $|\mathcal{J}_\infty \mathcal{X}|$  by the composite map

$$|\mathcal{J}_\infty \mathcal{X}| \xrightarrow{\varphi_{\infty,0}} |\mathcal{J}_0 \mathcal{X}| \xrightarrow{\text{sht}} \mathbb{Q}.$$

So we have a measurable function

$$\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} : |\mathcal{J}_\infty \mathcal{X}| \rightarrow \mathfrak{R}^{1/r}$$

for some sufficiently divisible positive integer  $r$ .

Next, we define the *order function* along an ideal. Let  $\mathcal{X}$  be a DM stack,  $\mathcal{Y}$  a closed substack of  $\mathcal{X}$ , and  $\mathcal{I}$  its ideal sheaf on  $\mathcal{X}$ . For  $\alpha \in |\mathcal{J}_\infty \mathcal{X}| \setminus |\mathcal{J}_\infty \mathcal{Y}|$ , we define a rational number  $\text{ord}_{\mathcal{I}} \alpha$  as follows. The twisted jet  $\alpha$  can be written by a morphism  $\alpha : [\text{Spec } \mathbb{C}[[t]]/\mu_l] \rightarrow \mathcal{X}$  for some  $l \geq 1$ . Hence we have the lift  $\bar{\alpha} : \text{Spec } \mathbb{C}[[t]] \rightarrow \mathcal{X}$  and have the induced ring homomorphism  $\bar{\alpha}^{-1} : \mathcal{O}_{\mathcal{X}} \rightarrow \mathbb{C}[[t]]$ . Let  $m$  be the integer satisfying  $\bar{\alpha}^{-1} \mathcal{I} = (t^m)$ . Then we define  $\text{ord}_{\mathcal{I}} \alpha := \frac{m}{l}$ .

Let  $f : \mathcal{X} \rightarrow X$  be a morphism from a DM stack to a variety. Then  $f$  induces a map  $f_\infty : |\mathcal{J}_\infty \mathcal{X}| \rightarrow |J_\infty X|$  as follows: For a twisted jet  $\mathcal{D}_{n,S}^l \rightarrow \mathcal{X}$ , we have the composition  $\mathcal{D}_{n,S}^l \rightarrow \mathcal{X} \rightarrow X$ . Since  $D_{n,S}$  is the coarse moduli space of  $\mathcal{D}_{n,S}^l$ , the above map uniquely factors through  $\mathcal{D}_{n,S}^l \rightarrow D_{n,S}$ , and we have an  $n$ -th jet  $D_{n,S} \rightarrow X$ . Since the map  $\mathcal{D}_{n,S}^l \rightarrow D_{n,S}$  is defined by  $t \mapsto t^l$ , the order functions on  $\mathcal{X}$  and  $X$  are compatible with  $f_\infty$ . That is, for an ideal sheaf  $I$  on  $X$ ,  $\text{ord}_I \circ f_\infty = \text{ord}_{f^* I}$  holds.

Here, we can state Yasuda's transformation rule for a proper birational morphism  $f : \mathcal{X} \rightarrow X$  from a smooth DM stack to a variety. The *Jacobian ideal sheaf* of  $f$  is defined to be the 0-th Fitting ideal sheaf of  $\Omega_{\mathcal{X}/X}$ , and we denote by  $\text{Jac}_f$ .

**Theorem 2.8** ([20]). *Let  $X$  be a variety,  $\mathcal{X}$  a smooth DM stack, and  $f : \mathcal{X} \rightarrow X$  a proper birational morphism. Let  $A$  be a subset of  $|J_\infty X|$  and  $F : A \rightarrow \mathfrak{R}^{1/r}$  a measurable function. Then  $F \circ f_\infty : f_\infty^{-1}(A) \rightarrow \mathfrak{R}^{1/r}$  is measurable, and*

$$\int_A F d\mu_X = \int_{f_\infty^{-1}(A)} (F \circ f_\infty) \mathbb{L}^{-\text{ord}_{\text{Jac}_f} + \mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}} \in \mathfrak{R}^{1/r} \cup \{\infty\}$$

holds.

**2.6. Minimal log discrepancies and jet schemes.** In [7], Ein, Mustașă, and Yasuda showed that mld's can be described by the language of jet schemes. Suppose  $X$  is a  $\mathbb{Q}$ -Gorenstein variety with Gorenstein index  $r$  and set  $n := \dim X$ . Then, we have a natural map

$$(\Omega_X^n)^{\otimes r} \longrightarrow \mathcal{O}_X(rK_X).$$

Since  $\mathcal{O}_X(rK_X)$  is an invertible sheaf, we have an ideal sheaf  $J_{r,X}$  such that the image of this map is  $J_{r,X} \mathcal{O}_X(rK_X)$ .

**Theorem 2.9** ([7]). *Let  $X$  be a normal,  $d$ -dimensional  $\mathbb{Q}$ -Gorenstein variety with Gorenstein index  $r$ ,  $Y_1, \dots, Y_s$  proper closed subschemes of  $X$ , and  $W$  a proper closed subset of  $X$ . If  $q_1, \dots, q_s$  are positive real numbers, then*

$$\text{mld}_W(X, \sum_{i=1}^s q_i Y_i) = \inf_{m \in \mathbb{N}^s} \left\{ d - \sum_{i=1}^s q_i m_i - \dim \int_{A_m} \mathbb{L}^{\frac{1}{r} \text{ord}_{J_{r,X}}} d\mu_X \right\},$$

where  $A_m := \varphi_{\infty,0}^{-1}(W) \cap \bigcap_{1 \leq i \leq s} \text{ord}_{Y_i}^{-1}(\geq m_i) \subset J_\infty X$ .

Moreover, if the mld is finite, then the infimum on the right-hand side is actually a minimum, and the minimum is attained by some  $m \in S$ , where  $S$  is a finite subset of  $\mathbb{N}^s$  and only depends on the numerical data of a resolution of  $(X, \sum_{i=1}^s q_i Y_i)$  and  $W$ . If the mld is infinite, then a negative value is attained by some  $m \in S$ .

*Remark 2.10.* The finite set  $S$  can be constructed as follows. Take a resolution  $\pi : X' \rightarrow X$  of the log pair  $(X, \sum_{i=1}^s q_i Y_i)$  and  $W$ . Let  $D_1, \dots, D_c$  be the divisors on  $X'$  satisfying  $\pi(D_j) \subset W$ . Then we can take  $S$  as

$$S = \{m = (\text{ord}_{D_j} Y_1, \dots, \text{ord}_{D_j} Y_s) \in \mathbb{N}^s \mid 1 \leq j \leq c\}.$$

### 3. LOWER SEMI-CONTINUITY PROBLEMS ON VARIETIES WITH QUOTIENT SINGULARITIES

**3.1. Minimal log discrepancies and twisted jet stacks.** Let  $X$  be a  $d$ -dimensional  $\mathbb{Q}$ -Gorenstein variety with  $\mathbb{Q}$ -Gorenstein index  $r$ ,  $\mathfrak{a}$  an  $\mathbb{R}$ -ideal sheaf, and  $W$  a closed subset of  $X$ . Assume that there exists a crepant resolution  $f : \mathcal{X} \rightarrow X$  in the category of DM stacks. In this setting, we can describe  $\text{mld}_W(X, \mathfrak{a})$  by a motivic integration on  $\mathcal{X}$ .

The  $\mathbb{R}$ -ideal sheaf  $\mathfrak{a}$  can be written by  $\mathfrak{a} = \prod_{i=1}^s \mathfrak{a}_i^{q_i}$  with ideal sheaves  $\mathfrak{a}_i$  and positive real numbers  $q_i$ . By Theorem 2.9,

$$\mathrm{mld}_W(X, \mathfrak{a}) = \inf_{m \in \mathbb{N}^s} \left\{ d - \sum_{i=1}^s q_i m_i - \dim \int_{A_{W,m}} \mathbb{L}^{\frac{1}{r} \mathrm{ord}_{J_{r,X}}} d\mu_X \right\},$$

where  $A_{W,m} := \varphi_{\infty,0}^{-1}(W) \cap \bigcap_{1 \leq i \leq s} \mathrm{ord}_{\mathfrak{a}_i}^{-1}(\geq m_i) \subset |J_{\infty} X|$ .

We apply Theorem 2.8 to the resolution  $f$ . We write  $\mathfrak{a}'_i := \mathfrak{a}_i \mathcal{O}_{\mathcal{X}}$  and denote by  $\varphi_{\infty,b}$  the composite map  $\mathcal{J}_{\infty} \mathcal{X} \xrightarrow{\varphi_{\infty,0}} \mathcal{J}_0 \mathcal{X} \xrightarrow{\varphi_{0,b}} \mathcal{X}$ .

**Lemma 3.1.** *Let  $X, \mathcal{X}, \mathfrak{a}, \mathfrak{a}'$  and  $W$  be as above. Then we have*

$$\mathrm{mld}_W(X, \mathfrak{a}) = \inf_{m \in \mathbb{N}^s} \left\{ d - \sum_{i=1}^s q_i m_i - \dim \int_{A'_{W,m}} \mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}} \right\},$$

where we denote

$$A'_{W,m} := (f \circ \varphi_{\infty,b})^{-1}(W) \cap \bigcap_{1 \leq i \leq s} \mathrm{ord}_{\mathfrak{a}'_i}^{-1}(\geq m_i) \subset |\mathcal{J}_{\infty} \mathcal{X}|.$$

*Proof.* First, we easily have  $f_{\infty}^{-1}(A_{W,m}) = A'_{W,m}$ .

By the definition of the Jacobian ideal  $\mathrm{Jac}_f$ , the image of the canonical morphism  $f^* \Omega_X^d \rightarrow \Omega_{\mathcal{X}}^d$  is equal to  $\mathrm{Jac}_f \otimes \Omega_{\mathcal{X}}^d$ . Let  $r$  be the Gorenstein index of  $X$ . By the definition of  $J_{r,X}$ , the image of the canonical morphism  $(\Omega_X^d)^{\otimes r} \rightarrow \mathcal{O}_X(rK_X)$  is equal to  $J_{r,X} \otimes \mathcal{O}_X(rK_X)$ . Therefore we have an equation  $\mathrm{Jac}_f^r = J_{r,X} \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/X})$ . Since  $f$  is crepant in this case, we have

$$\frac{1}{r} \mathrm{ord}_{J_{r,X}} \circ f_{\infty} = \mathrm{ord}_{\mathrm{Jac}_f}.$$

By Theorem 2.8, we can conclude

$$\int_{A_{W,m}} \mathbb{L}^{\frac{1}{r} \mathrm{ord}_{J_{r,X}}} d\mu_X = \int_{A'_{W,m}} \mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}},$$

which completes the proof.  $\square$

**3.2. Proof of Theorem 1.2.** We prove a more general statement.

**Theorem 3.2.** *Let  $f : X \rightarrow T$  be a flat surjective morphism between varieties,  $\mathfrak{A}$  an  $\mathbb{R}$ -ideal sheaf on  $X$ , and  $W$  a closed subset of  $X$ . Suppose that  $W$  is proper over  $T$  and each  $W_p$  is a proper subset of  $X_p$  and that each  $X_p$  is  $\mathbb{Q}$ -Gorenstein. In addition, assume that there exist a DM stack  $\mathcal{X}$  and a morphism  $g : \mathcal{X} \rightarrow X$  such that for each closed point  $p \in T$ , the induced morphism  $g_p : \mathcal{X}_p \rightarrow X_p$  is a crepant resolution.*

*Then, the function*

$$|T| \rightarrow \mathbb{R} \cup \{-\infty\}; \quad p \mapsto \mathrm{mld}_{W_p}(X_p, \mathfrak{A}_p)$$

*is lower semi-continuous.*

To prove this theorem, we borrow an idea from the argument in [16, Proposition 2.3]. We need two lemmas. First one is about the set  $S$  in Remark 2.10.

**Lemma 3.3.** *Let  $f : X \rightarrow T$ ,  $\mathfrak{A}$ , and  $W$  be as in Theorem 3.2. In addition, suppose that  $\mathfrak{A}_p \neq 0$  for any  $p \in |T|$ . Then, for each  $p \in |T|$ , we can take a finite set  $S_p \subset \mathbb{N}^s$  in Theorem 2.9 for  $(X_p, \mathfrak{A}_p)$  and  $W_p$  such that  $\bigcup_{p \in T} S_p$  is also a finite set.*

*Proof.* Take a log resolution  $g : X' \rightarrow X$  of  $(X, \mathfrak{A})$  and  $W$ . Let  $D = \bigcup D_i$  be the union of the exceptional divisors and the supports of  $\mathfrak{A}\mathcal{O}_{X'}$  and  $g^{-1}(W)$ . By generic smoothness, we can take a non-empty open set  $U \subset T$  such that  $X'$  and  $D_i$  are smooth over  $U$ . Take  $V \subset X$  be a non-empty open set over which  $g$  is isomorphic. Since  $f$  is flat and surjective,  $f(V)$  is also open. Then, for any  $p \in U \cap f(V)$ , the restriction to the fiber  $g_p : X'_p \rightarrow X_p$  is a log resolution of  $(X_p, \mathfrak{A}_p)$  and  $W_p$ . Hence, we can take  $S_p$  uniformly for  $p \in U \cap f(V)$ . Since  $U \cap f(V)$  is a non-empty open subset of  $T$ , the assertion follows from the induction on  $\dim T$ .  $\square$

Second lemma is about the shift function on a family. For a smooth morphism  $f : \mathcal{X} \rightarrow T$  from a DM stack to a variety, the relative 0-th twisted jet stack  $\mathcal{J}_0(\mathcal{X}/T)$  can be defined. An element of  $|\mathcal{J}_0(\mathcal{X}/T)|$  can be written by  $(p, \alpha)$  with  $p \in |T|$  and  $\alpha \in \mathcal{J}_0(\mathcal{X}_p)$ . On  $|\mathcal{J}_0(\mathcal{X}/T)|$ , the shift function  $\text{sht}$  can be defined by  $\text{sht}(p, \alpha) = \text{sht}(\alpha)$ .

**Lemma 3.4.** *Let  $f : \mathcal{X} \rightarrow T$  be as above. For a connected component  $\mathcal{V}$  of  $|\mathcal{J}_0(\mathcal{X}/T)|$ ,  $\text{sht}$  is constant on  $\mathcal{V}$ .*

*Proof.* We may assume that  $\mathcal{X}$  is a quotient stack  $[M/G]$ . Then the relative twisted jet stack can be written by

$$\mathcal{J}_0([M/G]/T) = \bigsqcup_{l \geq 1} \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_{nl}^{(a)}(M/T)/C_a].$$

Take  $l \geq 1$  and  $a : \mu_l \hookrightarrow G$  such that  $\mathcal{V} \subset [J_{nl}^{(a)}(M/T)/C_a]$ . Then, for every  $x \in |\mathcal{V}|$ ,  $a : \mu_l \hookrightarrow G$  factors through  $\text{Aut } x \subset G$ .

Let  $T_{\mathcal{X}/T}$  be the relative tangent bundle on  $\mathcal{X}$  over  $T$ , and  $i : \mathcal{V} \rightarrow \mathcal{X}$  the projection. Then  $\mu_l$  acts on  $i^*T_{\mathcal{X}/T}$  over  $T$ , and this action is fiberwise compatible with the  $\mu_l$ -action on  $T_{\mathcal{X}_x}$  in Section 2.5. This action induces a decomposition  $i^*T_{\mathcal{X}/T} = \bigoplus_{i=1}^l T_i$ , where  $T_i$  is the vector bundle on which  $\xi_l \in \mu_l$  acts by the multiplication of  $\xi_l^i$ . It follows that for each  $i$ ,  $\dim T_{x,i}$  is constant on  $|\mathcal{V}|$ , which completes the proof.  $\square$

*Proof of Theorem 3.2.* If  $W_p = \emptyset$ , then  $\text{mld}_{W_p}(X_p, \mathfrak{A}_p) = \infty$ . If  $W_p \neq \emptyset$  and  $\mathfrak{A}_p = 0$ , then  $\text{mld}_{W_p}(X_p, \mathfrak{A}_p) = -\infty$ . Since  $W$  is proper over  $T$ ,

$$T' := f(W) \cap \{p \in |T| \mid \mathfrak{A}_p = 0\}$$

is closed in  $|T|$ . Replacing  $T$  by  $T \setminus T'$ , we may assume that  $\mathfrak{A}_p \neq 0$  for any  $p \in |T|$ .

Since  $\mathfrak{A}$  is an  $\mathbb{R}$ -ideal sheaf, it can be written by  $\mathfrak{A} = \prod_{i=1}^s \mathfrak{A}_i^{q_i}$  with ideal sheaves  $\mathfrak{A}_i$  and positive real numbers  $q_i$ . We denote  $\mathfrak{A}'_i := \mathfrak{A}_i \mathcal{O}_{\mathcal{X}}$  and denote by  $(\mathfrak{A}'_i)_p$  the restriction of ideal  $\mathfrak{A}'_i$  to the fiber  $\mathcal{X}_p$ .

For  $p \in |T|$  and  $m \in \mathbb{N}^s$ , we put

$$B_{p,m} := (g_p \circ \varphi_{\infty,b}^{\mathcal{X}_p})^{-1}(W_p) \cap \bigcap_{1 \leq i \leq s} \text{ord}_{(\mathfrak{A}'_i)_p}^{-1}(\geq m_i).$$

Take a finite set  $S_p \subset \mathbb{N}^s$  for each closed point  $p \in T$  as in Lemma 3.3. Fix a multi-index  $m \in \bigcup_{p \in |X|} S_p$ . By Lemma 3.1, it is sufficient to prove that the function

$$|T| \rightarrow \mathbb{Q}; \quad p \mapsto \dim \int_{B_{p,m}} \mathbb{L}^{\mathfrak{s}_{\mathcal{X}_p}} d\mu_{\mathcal{X}_p}$$

is upper semi-continuous.

Let  $\mathcal{Y}_i$  be the closed substack of  $\mathcal{X}$  corresponding to the ideal sheaf  $\mathfrak{A}'_i$ . We identify the relative twisted jet stack  $\mathcal{J}_m(\mathcal{Y}_i/T)$  with a closed substack of  $\mathcal{J}_m(\mathcal{X}/T)$ , and we also identify  $B_{p,m}$  with a closed subset of  $\mathcal{J}_\infty(\mathcal{X}/T)_p$ .

For a connected component  $\mathcal{V}$  of  $|\mathcal{J}_0(\mathcal{X}/T)|$ , the shift function is constant on  $\mathcal{V}$  by Lemma 3.4. Since  $|\mathcal{J}_0(\mathcal{X}/T)|$  has finitely many connected components, it is sufficient to prove that the function

$$|T| \rightarrow \mathbb{Z}; \quad p \mapsto \dim \mu_{\mathcal{X}}((\varphi_{\infty,0}^{\mathcal{X}/T})^{-1}(\mathcal{V}) \cap B_{p,m})$$

is upper semi-continuous for each  $\mathcal{V}$ . Take a positive integer  $m'$  such that  $m' \geq m_i$  for any  $i$ . Since  $B_{p,m}$  is an  $m'$ -cylinder, we have

$$\mu_{\mathcal{X}}((\varphi_{\infty,0}^{\mathcal{X}/T})^{-1}(\mathcal{V}) \cap B_{p,m}) = \{S \cap (f \circ g \circ \varphi_{m',b}^{\mathcal{X}/T})^{-1}(p)\} \mathbb{L}^{-m'd},$$

where we put

$$\begin{aligned} S &:= (\varphi_{m',0}^{\mathcal{X}/T})^{-1}(\mathcal{V}) \cap (\varphi_{m',b}^{\mathcal{X}/T})^{-1}(g^{-1}(W)) \\ &\quad \cap \bigcap_{1 \leq i \leq s} (\varphi_{m',m_i}^{\mathcal{X}/T})^{-1}(|\mathcal{J}_{m_i}(\mathcal{Y}_i/T)|). \end{aligned}$$

Let  $F$  be the composite map

$$S \hookrightarrow |\mathcal{J}_{m'}(\mathcal{X}/T)| \xrightarrow{\varphi_{m',0}^{\mathcal{X}/T}} |\mathcal{J}_0(\mathcal{X}/T)| \xrightarrow{\varphi_{0,b}^{\mathcal{X}/T}} |\mathcal{X}| \xrightarrow{g} |X| \xrightarrow{f} |T|.$$

Then this theorem reduces to the upper semi-continuity property of the function

$$|T| \rightarrow \mathbb{Z}; \quad p \mapsto \dim F^{-1}(p).$$

For each integer  $n$ , we set

$$\begin{aligned} S_{\geq n} &:= \{s \in S \mid \dim_s F^{-1}(F(s)) \geq n\}, \\ |T|_{\geq n} &:= \{p \in |T| \mid \dim F^{-1}(p) \geq n\}. \end{aligned}$$

Then we have  $F(S_{\geq n}) = |T|_{\geq n}$ . We need to show that  $|T|_{\geq n}$  is closed in  $|T|$ , but we know only that  $S_{\geq n}$  is a closed subset of  $S$  by Lemma 2.4. The space  $|\mathcal{J}_{m'}(\mathcal{X}/T)|$  admits a  $\mathbb{C}^*$ -action (see Section 2.3). Each  $|\mathcal{J}_{m_i}(\mathcal{Y}_i/T)|$  is a  $\mathbb{C}^*$ -invariant closed subset of  $|\mathcal{J}_{m'}(\mathcal{X}/T)|$ , and each fibre of the morphism  $\varphi_{m',0}^{\mathcal{X}/T}$  is invariant on this action. Hence  $S_{\geq n}$  is a  $\mathbb{C}^*$ -invariant closed subset of  $|\mathcal{J}_{m'}(\mathcal{X}/T)|$ . By Lemma 2.7,  $\varphi_{m',0}^{\mathcal{X}/T}(S_{\geq n})$  is closed in  $|\mathcal{J}_0(\mathcal{X}/T)|$ . Since both of the maps  $\varphi_{0,b}^{\mathcal{X}/T}$  and  $g$  are closed, the set

$$|X|_{\geq n} := (g \circ \varphi_{0,b}^{\mathcal{X}/T})(\varphi_{m',0}^{\mathcal{X}/T}(S_{\geq n}))$$

is also closed in  $|X|$ . Because  $|X|_{\geq n}$  is a subset of  $|W|$  and  $W$  is proper over  $T$ , we can conclude the set

$$|T|_{\geq n} = f(|X|_{\geq n})$$

is also closed in  $|T|$ , which completes the proof.  $\square$

*Proof of Theorem 1.2.* We prove (i) first. Since  $X$  is  $\mathbb{Q}$ -Gorenstein, we may assume  $\Delta = 0$  by forcing  $\Delta$  to  $\mathfrak{a}$ . For  $i = 1, 2$ , we denote by  $p_i : X \times X \rightarrow X$  the  $i$ -th projection. Since  $\mathfrak{a}$  is an  $\mathbb{R}$ -ideal sheaf, it can be written by  $\mathfrak{a} = \prod_{i=1}^s \mathfrak{a}_i^{q_i}$  with  $\mathbb{R}$ -ideal sheaves  $\mathfrak{a}_i$  and positive real numbers  $q_i$ . Set  $\mathfrak{A} = \prod_{i=1}^s (p_1^* \mathfrak{a}_i)^{q_i}$ , and  $W$  be the diagonal set of  $X \times X$ . Applying Theorem 3.2 to the morphism  $p_2 : X \times X \rightarrow X$ , the ideal  $\mathfrak{A}$ , and the closed set  $W$ , we have the assertion in (i).

Next we prove (ii). Since  $X$  is  $\mathbb{Q}$ -Gorenstein, we may assume  $\Delta = 0$  by forcing  $\Delta \times T$  to  $\mathfrak{A}$ . Applying Theorem 3.2 to the projection  $X \times T \rightarrow T$ , the ideal  $\mathfrak{A}$ , and the closed set  $\{x\} \times T$ , we have the assertion in (ii).  $\square$

Now, we can prove Corollary 1.4. Matsushita proved the same statement when  $X$  is a smooth symplectic variety and  $D_0$  is a  $\mathbb{Q}$ -divisor [15, Proposition 2.1, Lemma 2.1]. First, by Theorem 1.2 (1), this can be extended to the case when  $X$  is a symplectic variety with only quotient singularities. In addition, Kawakita's result about ACC conjecture [10] can extend it to the case when  $D_0$  is an  $\mathbb{R}$ -divisor.

*Proof of Corollary 1.4.* Let  $Y$  be a normal variety and  $\Gamma$  be a finite subset of  $[0, 1]$ . Then, we define a set  $A(X, \Gamma)$  as

$$A(X, \Gamma) := \{\text{mld}_x(X, \Delta) \mid (X, \Delta) \text{ is a log canonical pair, and } \Delta \in \Gamma\},$$

here we write  $\Delta \in \Gamma$  if all coefficients of  $\Delta$  are in  $\Gamma$ .

Let  $\Gamma$  be the set of all coefficients of  $D_0$ . By [18], we need to show that the set  $\bigcup_{i \geq 0} A(X_i, \Gamma)$  satisfies the ascending chain condition and that LSC conjecture holds for each  $X_i$ .

By [17, Main Theorem, Proposition 2], since  $X_0$  has only terminal singularities,  $X_0$  and  $X_1$  can deform to a variety  $X'$  by locally trivial deformations at the same time. So, inductively, we can say that  $X_i$  has only quotient singularities and that  $A(X_i, \Gamma) = A(X, \Gamma)$ . By [10, Theorem 1.2], we already know that  $A(X, \Gamma)$  is finite. Hence  $\bigcup_{i \geq 0} A(X_i, \Gamma)$  is also finite. On the other hand, since  $X_i$  has only quotient singularities, LSC conjecture holds for each  $X_i$  by Theorem 1.2. Thus, we are done.  $\square$

#### 4. IDEAL-ADIC SEMI-CONTINUITY PROBLEM ON VARIETIES WITH A $\mathbb{C}^*$ -ACTION

**4.1. Proof of Proposition 1.6.** Before we start to prove Proposition 1.6, we provide some general remarks on Conjecture 1.5.

*Remark 4.1.* Let  $(X, \Delta, \mathfrak{a})$  and  $Z$  be as in Conjecture 1.5, and  $f : X' \rightarrow X$  a proper birational morphism from a  $\mathbb{Q}$ -Gorenstein variety  $X'$ . We denote  $\mathfrak{a}\mathcal{O}_{X'} := \prod_{i=1}^s (\mathfrak{a}_i \mathcal{O}_{X'})^{r_i}$ . Suppose  $\Delta' := f^*(K_X + \Delta) - K_{X'}$  is effective. Then, Conjecture 1.5 holds for  $(X, \Delta, \mathfrak{a})$  and  $Z$  if the conjecture holds for  $(X', \Delta', \mathfrak{a}\mathcal{O}_{X'})$  and  $f^{-1}(Z)$ .

This follows from that  $\text{mld}_Z(X, \Delta, \mathfrak{c}) = \text{mld}_{f^{-1}(Z)}(X', \Delta', \mathfrak{c}\mathcal{O}_{X'})$  holds for any  $\mathbb{R}$ -ideal sheaf  $\mathfrak{c}$  and that  $\mathfrak{a}_i + I_Z^l = \mathfrak{b}_i + I_Z^l$  implies  $\mathfrak{a}_i \mathcal{O}_{X'} + I_{f^{-1}(Z)}^l = \mathfrak{b}_i \mathcal{O}_{X'} + I_{f^{-1}(Z)}^l$ .

*Remark 4.2.* Let  $\prod_{i=1}^s \mathfrak{a}_i^{r_i}$  be an  $\mathbb{R}$ -ideal sheaf,  $f : X' \rightarrow X$  an extraction, and  $E$  a divisor on  $X'$ . If  $c_X(E) \subset Z$ , there exists a positive integer  $l_E$  such that: if ideal sheaves  $\mathfrak{b}_1, \dots, \mathfrak{b}_s$  satisfy  $\mathfrak{a}_i + I_Z^{l_E} = \mathfrak{b}_i + I_Z^{l_E}$ , then

$$a_E(X, \Delta, \prod_{i=1}^s \mathfrak{a}_i^{r_i}) = a_E(X, \Delta, \prod_{i=1}^s \mathfrak{b}_i^{r_i})$$

holds. In fact, if we take  $l_E$  such that  $\text{ord}_E I_Z^{l_E} > \text{ord}_E \mathfrak{a}_i$  for each  $i$ , then  $\text{ord}_E \mathfrak{a}_i = \text{ord}_E \mathfrak{b}_i$  holds, and this implies the above equation.

By this remark and Remark 2.1, the inequality

$$\text{mld}_Z(X, \Delta, \prod_{i=1}^s \mathfrak{a}_i^{r_i}) \geq \text{mld}_Z(X, \Delta, \prod_{i=1}^s \mathfrak{b}_i^{r_i})$$

in Conjecture 1.5 is obvious.

Let  $X = \text{Spec } A$  be an affine variety with a  $\mathbb{C}^*$ -action, and  $A = \bigoplus_{m \in \mathbb{Z}} A^{(m)}$  be the induced graded ring structure, where

$$A^{(m)} := \{f \in A \mid \gamma \cdot f = \gamma^m f \text{ for any } \gamma \in \mathbb{C}^*\}.$$

In what follows, we assume that the  $\mathbb{C}^*$ -action is of ray type, and  $A = \bigoplus_{m \geq 0} A^{(m)}$ .

For an element  $f \in A$ , we get the unique expression  $f = \sum_{m \geq 0} f^{(m)}$  with  $f^{(m)} \in A^{(m)}$ . Then we define  $\deg f$  and  $f \circ t$  as follows:

$$\deg f := \min\{m \mid f^{(m)} \neq 0\}, \quad f \circ t := \sum_m t^m f^{(m)} \in A[t].$$

For an ideal  $I \subset A$ , we define  $\tilde{I} \subset A[t]$  as

$$\tilde{I} := \text{ideal} \left( \frac{f \circ t}{t^{\deg f}} \mid f \in I \setminus \{0\} \right).$$

For  $\gamma \in \mathbb{C}$ , the restriction  $\tilde{I}_\gamma$  is defined as the ideal

$$\tilde{I}_\gamma := \{f(\gamma) \mid f(t) \in \tilde{I}\} \subset A.$$

It is clear that  $\tilde{I}_1 = I$ . For  $\gamma \in \mathbb{C}^*$ , we have the ring automorphism

$$\phi_\gamma : A \longrightarrow A; \quad \sum_m f^{(m)} \mapsto \sum_m \gamma^m f^{(m)},$$

and  $\phi_\gamma(\tilde{I}_1) = \tilde{I}_\gamma$  holds.

*Proof of Proposition 1.6.* Since  $X$  is  $\mathbb{Q}$ -Gorenstein, we may assume  $\Delta = 0$  by forcing  $\Delta$  to  $\prod_{i=1}^s \mathfrak{a}_i^{r_i}$ . By exchanging  $t$  with  $t^{-1}$  in  $\mathbb{C}^* = \text{Spec } \mathbb{C}[t, t^{-1}]$ , we may assume  $A = \bigoplus_{m \geq 0} A^{(m)}$ . Since  $Z$  is the set of all  $\mathbb{C}^*$ -fixed points, it follows that  $I_Z = \bigoplus_{m \geq 1} A^{(m)}$ . Since ideals  $\mathfrak{a}_1, \dots, \mathfrak{a}_s$  are  $\mathbb{C}^*$ -invariant, they are homogeneous. Let  $f_{i1}, \dots, f_{ik_i}$  be elements in  $A$  such that they generate  $\mathfrak{a}_i$ . Set  $l_i := 1 + \max_{1 \leq j \leq k_i} \{\deg f_{ij}\}$ .

By Remark 4.2, it is sufficient to show that if ideal sheaves  $\mathfrak{b}_1, \dots, \mathfrak{b}_s$  satisfy  $\mathfrak{a}_i + I_Z^{l_i} = \mathfrak{b}_i + I_Z^{l_i}$ , then

$$\text{mld}_Z(X, \prod_{i=1}^s \mathfrak{a}_i^{r_i}) \leq \text{mld}_Z(X, \prod_{i=1}^s \mathfrak{b}_i^{r_i})$$



holds.

We first prove that  $\mathfrak{a}_i \subset (\tilde{\mathfrak{b}}_i)_0$ . Since  $\mathfrak{a}_i = \text{ideal}(f_{i1}, \dots, f_{ik_i}) \subset \mathfrak{b}_i + I_Z^{l_i}$ , there exists  $h_{ij}$  for each  $1 \leq j \leq k_i$  such that

$$f_{ij} + h_{ij} \in \mathfrak{b}_i, \quad h_{ij} \in I_Z^{l_i}.$$

In respect to the degrees, we have  $\deg f_{ij} \leq l_i - 1 < \deg h_{ij}$ . Since  $f_{ij}$  is homogeneous, we have  $f_{ij} \in (\tilde{\mathfrak{b}}_i)_0$ . We thus get  $\mathfrak{a}_i \subset (\tilde{\mathfrak{b}}_i)_0$ .

This inclusion implies the inequality

$$\text{mld}_Z(X, \prod_{i=1}^s \mathfrak{a}_i^{r_i}) \leq \text{mld}_Z(X, \prod_{i=1}^s (\tilde{\mathfrak{b}}_i)_0^{r_i}).$$

Because  $(\tilde{\mathfrak{b}}_i)_1 = \mathfrak{b}_i$ , it is sufficient to show that

$$\text{mld}_Z(X, \prod_{i=1}^s (\tilde{\mathfrak{b}}_i)_0^{r_i}) \leq \text{mld}_Z(X, \prod_{i=1}^s (\tilde{\mathfrak{b}}_i)_1^{r_i}).$$

On the other hand, we have the ring automorphism  $\phi_\gamma : A \rightarrow A$  for  $\gamma \in \mathbb{C}^*$ . By the definition of  $\tilde{\mathfrak{b}}_i$ , we have  $\phi_\gamma((\tilde{\mathfrak{b}}_i)_1) = ((\tilde{\mathfrak{b}}_i)_\gamma)$ . Hence we have

$$\text{mld}_Z(X, \prod_{i=1}^s (\tilde{\mathfrak{b}}_i)_1^{r_i}) = \text{mld}_Z(X, \prod_{i=1}^s (\tilde{\mathfrak{b}}_i)_\gamma^{r_i})$$

for every  $\gamma \in \mathbb{C}^*$ . Therefore, the proof can be completed by showing that the function

$$|\mathbb{A}^1| \rightarrow \mathbb{R} \cup \{-\infty\}; \quad p \mapsto \text{mld}_Z(X, \prod_{i=1}^s (\tilde{\mathfrak{b}}_i)_p^{r_i})$$

is lower semi-continuous.

In order to prove the semi-continuity, we consider the relative twisted jet stacks. Take a  $\mathbb{C}^*$ -equivariant crepant resolution  $\mathcal{X} \rightarrow X$  and fix a positive integer  $l$  and  $m \in \mathbb{N}^s$ . Let  $\mathcal{Y}_i \subset \mathcal{X} \times \mathbb{A}^1$  be the closed substack corresponding to  $\tilde{\mathfrak{b}}_i \mathcal{O}_{\mathcal{X} \times \mathbb{A}^1}$ . Take a positive integer  $m'$  such that  $m' \geq m_i$  holds for any  $i$ . In addition, fix a connected component  $\mathcal{V}$  of  $|\mathcal{J}_0(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)|$ . Consider the following twisted jet stacks and morphisms

$$|\mathcal{J}_{m'}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)| \xrightarrow{\varphi_{m',0}} |\mathcal{J}_0(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)| \xrightarrow{\varphi_{0,b}} |\mathcal{X} \times \mathbb{A}^1| \xrightarrow{g} |X \times \mathbb{A}^1| \xrightarrow{f} |\mathbb{A}^1|,$$

where  $f$  is the second projection and  $g$  is the morphism induced by the resolution  $\mathcal{X} \rightarrow X$ . Then, we set

$$S := \varphi_{m',0}^{-1}(\mathcal{V}) \cap \varphi_{m',b}^{-1}(g^{-1}(W \times \mathbb{A}^1)) \cap \bigcap_{1 \leq i \leq s} \varphi_{m',m_i}^{-1}(|\mathcal{J}_{m_i}(\mathcal{Y}_i/\mathbb{A}^1)|).$$

Let  $F$  be the composite map  $S \hookrightarrow |\mathcal{J}_{m'}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)| \rightarrow |\mathbb{A}^1|$ . Then, as in the proof of Theorem 3.2, the assertion can reduce to the upper semi-continuity property of the function

$$|\mathbb{A}^1| \rightarrow \mathbb{Z}; \quad p \mapsto \dim F^{-1}(p).$$

For each integer  $n$ , we set

$$S_{\geq n} := \{s \in S \mid \dim_s F^{-1}(F(s)) \geq n\},$$

$$|\mathbb{A}^1|_{\geq n} := \{p \in |\mathbb{A}^1| \mid \dim F^{-1}(p) \geq n\}.$$

Then we have  $F(S_{\geq n}) = |\mathbb{A}^1|_{\geq n}$ .

By the same reason as in the proof of Theorem 3.2, it follows that  $(g \circ \varphi_{m',b})(S_{\geq n})$  is closed in  $|X \times \mathbb{A}^1|$ . By definition,  $\mathfrak{b}_i$  is isotrivial over  $\mathbb{A}^1 \setminus \{0\} = \mathbb{C}^*$ . In addition, we note that  $(g \circ \varphi_{m',b})(S_{\geq n})$  is contained in  $|W \times \mathbb{A}^1|$  and that  $W \subset X$  consists of  $\mathbb{C}^*$ -fixed points. Hence, if  $(p, \gamma) \in (g \circ \varphi_{m',b})(S_{\geq n})$  for some  $p \in |X|$  and  $\gamma \in \mathbb{A}^1 \setminus \{0\}$ , then  $(p, \gamma') \in (g \circ \varphi_{m',b})(S_{\geq n})$  for any  $\gamma' \in \mathbb{A}^1 \setminus \{0\}$ . Since  $(g \circ \varphi_{m',b})(S_{\geq n})$  is closed, the latter condition implies  $(p, 0) \in (g \circ \varphi_{m',b})(S_{\geq n})$ . Therefore we can conclude that  $F(S_{\geq n})$  is one of  $\emptyset$ ,  $\{0\}$  or  $\mathbb{A}^1$ , which completes the proof.  $\square$

**4.2. Toric case.** If  $(X, \Delta, \prod_{i=1}^s \mathfrak{a}_i^{r_i})$  is a toric triple and  $Z$  is a torus invariant closed set, then we can construct a  $\mathbb{C}^*$ -action on  $X$  as in Proposition 1.6.

*Proof of Theorem 1.8.* By Remark 2.2, we may assume that  $X$  is an affine toric variety and  $Z$  is irreducible. In addition, By Remark 4.1, we may assume that  $X$  is an affine  $\mathbb{Q}$ -factorial toric variety. Note that a  $\mathbb{Q}$ -factorial toric variety has a crepant resolution in the category of smooth DM toric stacks. As in the proof of Proposition 1.6, we may assume  $\Delta = 0$ .

Let  $n$  be the dimension of  $X$ ,  $M$  a free  $\mathbb{Z}$ -module of rank  $n$ , and set  $M_{\mathbb{R}} := M \otimes \mathbb{R}$ . Since  $X$  is an affine normal toric variety, we may assume  $X = \text{Spec } \mathbb{C}[\sigma \cap M]$  for some rational convex cone  $\sigma \subset M_{\mathbb{R}}$ . For  $m \in \sigma \cap M$ , we denote by  $\chi^m$  the corresponding function in  $\mathbb{C}[\sigma \cap M]$ . We call such an element a *monomial*. Since  $\mathfrak{a}_i$  and  $I_Z$  are torus invariant ideals, they are generated by monomials in  $\mathbb{C}[\sigma \cap M]$ .

If  $Z = \emptyset$ , then the assertion is trivial. In what follows, we assume  $Z \neq \emptyset$ . Since  $Z$  is torus invariant, there exists a face  $F$  of  $\sigma$  such that  $I_Z$  is generated by the monomials  $\chi^m$  for  $m \in (\sigma \setminus F) \cap M$ . Since  $Z \neq \emptyset$ , we can take a hyperplane  $H$  such that  $\sigma \cap H = F$ . Hence there exists  $w \in M^{\vee}$  such that  $\langle m, w \rangle \geq 1$  for every  $m \in (\sigma \setminus F) \cap M$  and that  $\langle m, w \rangle = 0$  for every  $m \in F$ . We fix such  $w \in M^{\vee}$ .

The element  $w \in M^{\vee}$  provides the  $\mathbb{C}^*$ -action on the ring  $\mathbb{C}[\sigma \cap M]$  as follows: for  $\gamma \in \mathbb{C}^*$

$$\mathbb{C}[\sigma \cap M] \longrightarrow \mathbb{C}[\sigma \cap M]; \quad \chi^m \mapsto \gamma^{\langle m, w \rangle} \chi^m.$$

Then, this  $\mathbb{C}^*$ -action is of ray type because  $\langle m, w \rangle \geq 0$  for every  $m \in \sigma \cap M$ . In addition,  $Z$  is the set of all  $\mathbb{C}^*$ -fixed points because  $I_Z = \bigoplus_{m \geq 1} A^{(m)}$ . Since  $\mathfrak{a}_1, \dots, \mathfrak{a}_s$  are torus invariant,  $\mathfrak{a}_1, \dots, \mathfrak{a}_s$  are  $\mathbb{C}^*$ -invariant. Therefore we can apply Proposition 1.6 to the toric pair and complete the proof.  $\square$

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1  
KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN.  
E-mail address: [nakamura@ms.u-tokyo.ac.jp](mailto:nakamura@ms.u-tokyo.ac.jp)